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A new perturbation theory for the dynamics of the Little–Hopfield model

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Abstract. In order to study the dynamical behaviour of the Little–Hopfield model of a neural network, we demonstrate a new perturbation theory for the retrieval of memory. This theory is characterized by a time-dependent perturbation variable in powers of which are calculated the overlap $m(t)$ as well as other order parameters such as the non-retrieval parameter $r(t)$, t being time. The second-order approximation in our perturbation scheme shows that, as time increases, the trajectories of the model system represented in the $m(t)$–$r(t)$ plane converge onto a curve which is identical to the so-called '*freezing*' line attained through the replica symmetric solution. In the course of developing our perturbation theory we generalize the exact approach proposed by Gardner *et al*. We also re-phrase previous approaches such as the theory by Amari and Maginu and by Coolen and Sherrington within the scheme of our perturbation theory.

1. Introduction

The complex dynamical behaviour of fully connected attractor networks [1, 2] have been the subjects of growing interest in recent years. Among others, the Little–Hopfield model [3, 4] for auto-associative memory has been investigated intensely as a prototype of such random frustrated systems. Although there are quite a number of approaches [5–15] to the dynamics of this model, the difficulties included in these approaches are rather serious.

Gardner *et al* [5] proposed the exact approach in which the formal expressions for the time-dependent order parameters are given on the basis of the path integral formulation. They showed that the number of such order parameters required to describe the dynamics increases with the time step. However, these formal expressions are extremely intricate and only solutions up to the second time step were obtained. An alternative approach by Amari and Maginu [7] (hereafter referred to as AM) is based on a crucial assumption that the noise term involved in the local alignment field for each neuron is a random Gaussian variable. On the basis of this assumption the recursion relations are obtained which give the time evolutions of the two order parameters of interest, i.e. the overlap and the variance of the Gaussian distribution. The qualitative properties of this model are fairly well reproduced by these recursion relations. However, it was shown by the computer simulations performed by Nishimori and Ozeki [12] that the above-described assumption is not fulfilled, at least when the retrieval of the memory fails. Coolen and Sherrington [13] (hereafter referred to as CS) followed the line of the AM approach further by calculating the probability distribution of the noise term using the replica technique. Their approach also starts from the assumption that the distribution function depends only on two order

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parameters, but there is no concrete foundation for this assumption, just as there is no guarantee for the above-described assumption underlying the AM theory.

With this situation, it is important to develop a theory keeping as far away from such assumptions as possible.

In this paper we propose a novel approach on the basis of the perturbation theory. Making use of Gardner's exact expressions for the order parameters, we take perturbation expansions using a time-dependent perturbation parameter. The recursion relations for the relevant order parameters are obtained without introducing any assumption in the first-order approximation, and the necessity of using an assumption arises only in the second-order approximation. It is also worth mentioning that the extent of our approach, i.e. the region of validity of the perturbation expansion, is found from the theoretical scheme itself.

This paper is organized as follows. In section 2 we give the outline of the Little–Hopfield model and define the important order parameters, i.e. the *overlap* $m(t)$ between the state of the system at time t and the pattern to be retrieved, and the '*non-retrieval parameter*' $r(t)$. In section 3 we generalize the exact approach due to Gardner *et al* and derive explicit expressions for some of the order parameters, including $r(t)$, at any time step t. We also evaluate the exact forms of noise distributions for the first few time steps. This generalization of the exact approach opens a path towards the establishment of a perturbation theory which we fully explain in section 4. First, we introduce a time-dependent perturbation parameter. The extent of our theory is derived from the condition that this parameter stays small at each time step. We see that this condition is fulfilled when the retrieval of the pattern is successful (hereafter to be referred to as the 'retrieval case'). Then, we evaluate the time development of the system in the first-order approximation and show that the AM theory is consistent with this first-order approximation in our theory. Furthermore, the probability distribution of the noise term turns out to be the Gaussian function as is assumed in the AM theory. We also calculate the time evolution of the system in the second-order approximation, from which we can see that the discrepancy between our theory and the AM theory appears in this second-order approximation. Our theory predicts that the trajectories represented in the $m(t)$–$r(t)$ plane converge onto the so-called '*freezing*' line determined by the replica symmetric analysis of CS.

In section 5 we show that the results of our simulations lend support to the above-mentioned prediction of our perturbation theory. Summary and discussions are given in section 6.

2. The Little–Hopfield model

The Little–Hopfield model system [3, 4] consists of N formal neurons, each of which is described by an Ising variable $S_i(t)$ ($i = 1, 2, \ldots, N$). We adopt the zero-temperature parallel dynamics; namely, at each step t, all neurons update their states deterministically according to the rule

$$S_i(t) = \mathrm{sgn}(h_i(t-1)) = \mathrm{sgn}\left(\sum_{j \neq i} J_{ij} S_j(t-1)\right) \tag{1}$$

where $h_i(t)$ is the local alignment field at time t. The strength of the interaction J_{ij} between neurons i and j is determined from a set of αN patterns $\{\xi_i^\mu\}$ ($\mu = 1, 2, \ldots, \alpha N$) according to the Hebb rule

$$J_{ij} = \frac{1}{N} \sum_{\mu=1}^{\alpha N} \xi_i^\mu \xi_j^\mu . \tag{2}$$

Each component ξ_i^μ of the pattern μ is a quenched random variable taking $+1$ or -1 with equal probability.

One of the most relevant order parameters in this model is the overlap $m(t) = m^\nu(t)$ between the state of the system at time t and the configuration of the pattern ν to be retrieved $m(t)$ being defined by

$$m(t) = m^\nu(t) = \frac{1}{N}\sum_{i=1}^{N} \xi_i^\nu S_i(t). \tag{3}$$

From equation (1) the time evolution of this variable is derived as

$$m(t) = \frac{1}{N}\sum_{i=1}^{N} \xi_i^\nu \operatorname{sgn}(m(t-1) + z_i(t-1)) \tag{4}$$

in which $z_i(t) \equiv \xi_i^\nu \frac{1}{N}\sum_{j\neq i}\sum_{\mu\neq\nu} \xi_i^\mu \xi_j^\mu S_j(t)$ is the noise term. The assumption of the AM theory [7], as well as that as the CS theory [13], is made for the probability distribution of this noise term.

Another important order parameter which has been discussed in the literature is the '*non-retrieval parameter*' $r(t)$:

$$r(t) = \frac{1}{\alpha N}\sum_{\mu\neq\nu}\{m^\mu(t)\}^2 = \frac{1}{\alpha N}\sum_{\mu\neq\nu}\left\{\frac{1}{N}\sum_{i=1}^{N}\xi_i^\mu S_i(t)\right\}^2. \tag{5}$$

The $r(t)$ parameter reflects the interference effects of non-retrieved patterns.

Now our theme is to evaluate the dynamical behaviours of these order parameters from the microscopic rule described by equation (1).

3. Generalization of the exact approach

To begin with, we try to generalize and improve the exact approach due to Gardner *et al* [5]. The generating function in the exact approach is defined by

$$Y[m(0)] = \left\langle \operatorname{Tr}\delta\left(m(0) - \frac{1}{N}\sum_i \xi_i^\nu S_i(0)\right)\prod_{i,t\geqslant 0}\theta(S_i(t+1)\operatorname{sgn}[h_i(t)])\right\rangle_\xi \tag{6}$$

where $m(0)$ is the initial overlap and Tr denotes the trace over the spin variable $S_i(t) = \pm 1$ ($i = 1, 2, \ldots, N$, $t = 0, 1, 2, \ldots$). The angular bracket refers to the average over the possible realization of the αN random patterns $\{\xi_i^\nu\}$. The symbols $\theta(x)$ and $\delta(x)$ respectively indicate the step function ($\theta(x) = 0$ for $x < 0$ and $\theta = 1$ for $x > 0$) and the Dirac δ function. Equation (6) is expressed in terms of the integral form

$$Y[m(0)] = \int\frac{\mathrm{d}\Lambda_0}{2\pi/N}\int\prod_{t\geqslant 1}\frac{\mathrm{d}\Lambda_t\,\mathrm{d}M_t}{2\pi/N}\int\prod_{0\leqslant t\leqslant t'}\frac{\mathrm{d}l_{tt'}\,\mathrm{d}q_{tt'}}{2\mathrm{i}/\alpha N}\int\prod_{tt'\geqslant 0}\frac{\mathrm{d}k_{tt'}\,\mathrm{d}v_{tt'}}{2\pi/\alpha N}\int\prod_{0\leqslant t\leqslant t'}\frac{\mathrm{d}p_{tt'}\,\mathrm{d}r_{tt'}}{2\pi/\alpha N}$$
$$\times\exp\left(\mathrm{i}N\lambda_0 m(0) + \mathrm{i}N\sum_{t\geqslant 1}\Lambda_t M_t + \mathrm{i}N\alpha C + \alpha N\ln W + N\ln Z\right) \tag{7}$$

where C, W and Z are the functions of the integration variables which appear in equation (7). The explicit forms of these functions are defined as follows:

$$C = \sum_{0\leqslant t\leqslant t'} l_{tt'}q_{tt'} + \sum_{tt'\geqslant 0} k_{tt'}v_{tt'} + \sum_{0\leqslant t\leqslant t'} p_{tt'}r_{tt'} + \mathrm{i}\sum_{t\geqslant 0}\frac{l_{tt}}{2} \tag{8}$$

$$W = \int\prod_{t\geqslant 0}\frac{\mathrm{d}n_t\,\mathrm{d}m_t}{2\pi}\exp\left(\mathrm{i}N\sum_{t\geqslant 0} n_t m_t - \mathrm{i}N\sum_{0\leqslant t\leqslant t'} q_{tt'}n_t n_{t'}\right)$$

$$\times \exp\left(-\mathrm{i}N\sum_{tt'\geqslant 0} v_{tt'}n_t m_{t'} - \mathrm{i}N\sum_{0\leqslant t\leqslant t'} p_{tt'}m_t m_{t'}\right) \tag{9}$$

$$Z = \mathrm{Tr}'\int_0^\infty \prod_{t\geqslant 0}\frac{\mathrm{d}\lambda_t}{\sqrt{2\pi}}\int_{-\infty}^\infty \prod_{t\geqslant 0}\frac{\mathrm{d}x_t}{\sqrt{2\pi}}\mathrm{e}^{-\mathrm{i}\Lambda_0 S(0)}$$
$$\times \exp\left[-\mathrm{i}\sum_{t\geqslant 0}\Lambda_t S(t) + \mathrm{i}\sum_{t\geqslant 0} x_t(\lambda_t - S(t+1)M_t + \alpha S(t)S(t+1))\right]$$
$$\times \exp\left[-\alpha\sum_{0\leqslant t\leqslant t'} r_{tt'}x_t x_{t'}S(t+1)S(t'+1) - \frac{\alpha}{2}\sum_{t\geqslant 0} r_{tt}x_t^2\right]$$
$$\times \exp\left[-\alpha\sum_{tt'\geqslant 0} k_{tt'}x_t S(t)S(t'+1) - \alpha\sum_{0\leqslant t\leqslant t'} l_{tt'}S(t+1)S(t'+1)\right] \tag{10}$$

where Tr' denotes the trace over the spin variable without suffix $S(t) = \pm 1$ ($t = 0, 1, 2, \ldots$).

The integrations are performed by making use of the steepest decent method; consequently, the saddle-point equation is obtained for each of the integration variables in the following form:

$$M_t = \langle S(t)\rangle_Z \qquad (t \geqslant 1) \tag{11}$$
$$\mathrm{i}q_{tt'} = \langle S(t)S(t')\rangle_Z \qquad (t \leqslant t') \tag{12}$$
$$\mathrm{i}v_{tt'} = \langle S(t)S(t'+1)x_{t'}\rangle_Z \qquad (t \geqslant 0, t' \geqslant 0) \tag{13}$$
$$\Lambda_t = \langle S(t+1)x_t\rangle_Z \qquad (t \geqslant 0) \tag{14}$$
$$\begin{aligned}\mathrm{i}p_{tt'} &= \langle S(t+1)x_t S(t'+1)x_{t'}\rangle_Z \qquad (t < t')\\ \mathrm{i}p_{tt'} &= \tfrac{1}{2}\langle x_t^2\rangle_Z \qquad (t = t')\end{aligned} \tag{15}$$
$$k_{tt'} = N\langle n_t^\mu m_{t'}^\mu\rangle_W \qquad (t \geqslant 0, t' \geqslant 0) \tag{16}$$
$$r_{tt'} = M\langle m_t^\mu m_{t'}^\mu\rangle_W \qquad (t \leqslant t') \tag{17}$$
$$l_{tt'} = N\langle n_t^\mu n_{t'}^\mu\rangle_W \qquad (t \leqslant t') \tag{18}$$

where $\langle A\rangle_W$ and $\langle A\rangle_Z$ (e.g. $A = S(t)$ for M_t, $A = n_t^\mu m_{t'}^\mu$ for $k_{tt'}$) denote the 'average' with respect to W of equation (9) and Z of equation (10) respectively, defined by

$$\langle A\rangle_W = \int_{-\infty}^\infty \prod_{t\geqslant 0}\frac{\mathrm{d}n_t}{\sqrt{2\pi}}\int_{-\infty}^\infty \prod_{t\geqslant 0}\frac{\mathrm{d}m_t}{\sqrt{2\pi}} A\exp\left(\mathrm{i}N\sum_{t\geqslant 0} n_t m_t - \mathrm{i}N\sum_{0\leqslant t\leqslant t'} q_{tt'}n_t n_{t'}\right)$$
$$\times \exp\left(-\mathrm{i}N\sum_{tt'\geqslant 0} v_{tt'}n_t m_{t'} - \mathrm{i}N\sum_{0\leqslant t\leqslant t'} p_{tt'}m_t m_{t'}\right)\Big/ W \tag{19}$$

$$\langle A\rangle_Z = \mathrm{Tr}'\int_0^\infty \prod_{t\geqslant 0}\frac{\mathrm{d}\lambda_t}{\sqrt{2\pi}}\int_{-\infty}^\infty \prod_{t\geqslant 0}\frac{\mathrm{d}x_t}{\sqrt{2\pi}} A\mathrm{e}^{-\mathrm{i}\Lambda_0 S(0)}$$
$$\times \exp\left[-\mathrm{i}\sum_{t\geqslant 1}\Lambda_t S(t) + \mathrm{i}\sum_{t\geqslant 0} x_t(\lambda_t - S(t+1)M_t + \alpha S(t)S(t+1))\right]$$
$$\times \exp\left[-\alpha\sum_{0\leqslant t\leqslant t'} r_{tt'}x_t x_{t'}S(t+1)S(t'+1) - \frac{\alpha}{2}\sum_{t\geqslant 0} r_{tt}x_t^2\right]$$
$$\times \exp\left[-\alpha\sum_{tt'\geqslant 0} k_{tt'}x_t S(t)S(t'+1) - \alpha\sum_{0\leqslant t<t'} l_{tt'}S(t+1)S(t'+1)\right]\Big/ Z. \tag{20}$$

From the definition of the generating function, it was shown by Gardner *et al* [5] that the overlap at time t is given as the solution of the saddle-point equation for M_t:

$$m(t) = M_t. \tag{21}$$

In a similar way the relation

$$r(t) = r_{tt} \tag{22}$$

holds for the non-retrieval parameter $r(t)$. It is easily seen that if we are to calculate the right-hand side of equations (21) and (22) explicitly, all the solutions of the saddle-point equations up to the $(t-1)$th step must be evaluated. Accordingly, all of the non-zero solutions of the saddle-point equations for the integration variables must be regarded as the order parameters as well. This means that the number of order parameters increases according to the increase of the time step.

Table 1. The exact solutions of saddle-point equations for order parameters.

Order parameter	Suffix	Gardner *et al*	Present work
M_t			
$\mathrm{i}q_{tt'}$	$t=t'$	$\frac{1}{2}$	
	$t<t'$		
$\mathrm{i}v_{tt'}$	$t\leqslant t'$	0	
	$t>t'+1$		
Λ_t	$t=0$	$\mathrm{i}\tanh^{-1}(m(0))$	
	$t\neq 0$	0	
$\mathrm{i}p_{tt'}$		0	
	$t<t'$		equation (23)
$k_{tt'}$	$t=t'$		i
	$t>t'$	0	
$r_{tt'}$	$t=t'$		equation (24)
	$t<t'$		equation (25)
$l_{tt'}$		0	

In table 1 we show a list of the parameters; some of them were previously obtained by Gardner *et al* as shown. In addition to these parameters we find that there are some parameters which have saddle-point equations that are explicitly solved, i.e. $k_{tt'}$ and $r_{tt'}$. Note that both of these parameters have the saddle-point equations of type (19). The integrals involved in these saddle-point equations are carried out by using the Gauss integral $\int_{-\infty}^{\infty}\mathrm{d}t\,\mathrm{e}^{-t^2}=\sqrt{\pi}$ and the following expressions are obtained:

$$k_{tt'} = \mathrm{i}V_{tt'} \qquad (t\leqslant t') \tag{23}$$

$$r_{tt} = 2\mathrm{i}\sum_{t_1\leqslant t_2}^{t} q_{t_1t_2}V_{t_1t}V_{t_2t} \tag{24}$$

$$r_{tt'} = r_{tt}V_{tt'} + \mathrm{i}\sum_{t_1\leqslant t_2}^{t'} q_{t_1t_2}\left(V_{t_1t}L_{t_2t'}^{(t)} + V_{t_2t}L_{t_1t'}^{(t)}\right) \qquad (t<t') \tag{25}$$

in which $V_{t_1t_2}$ and $L_{t_1t_2}^{(t_3)}$ are defined by

$$\begin{aligned} V_{t_1t_2} &= \sum_{t=t_1}^{t_2} v_{tt_1}V_{tt_2} \qquad (t_1<t_2) \\ V_{t_1t_2} &= 1 \qquad (t_1=t_2) \\ V_{t_1t_2} &= 0 \qquad (t_1>t_2) \end{aligned} \tag{26}$$

and

$$
\begin{aligned}
&L^{(t_3)}_{t_1t_2} = \sum_{t>t_3}^{t_2} v_{tt_1} V_{tt_2} + \sum_{t>t_1}^{t_3} v_{tt_1} L^{(t_3)}_{tt_2} \qquad (t_3 < t_2, t_1 < t_2, t_1 \neq t_3) \\
&L^{(t_3)}_{t_1t_2} \equiv 1 \qquad (t_3 < t_2, t_1 = t_2) \\
&L^{(t_3)}_{t_1t_2} \equiv 0 \qquad \text{(otherwise)}.
\end{aligned} \tag{27}
$$

We emphasize that the important order parameter $r(t) = r_{tt}$ is formulated for the first time in this paper.

From table 1 we see that only three kinds of parameters, M_t, $q_{tt'}$ and $v_{tt'}$ for $t > t'$, are left unsolved. The saddle-point equations (11)–(13) for these parameters have similar forms and become simple by the use of our results $k_{tt} = \mathrm{i}$, i.e.

$$
\begin{aligned}
\langle A \rangle_Z \propto \mathrm{Tr}' \frac{1 + m(0)S(0)}{2} \int_0^\infty \prod_{t\geqslant 0} \frac{\mathrm{d}\lambda_t}{\sqrt{2\pi}} \int_{-\infty}^\infty \prod_{t\geqslant 0} \frac{\mathrm{d}x_t}{\sqrt{2\pi}} A \\
\times \exp\left[+ \mathrm{i} \sum_{t\geqslant 0} x_t(\lambda_t - S(t+1)M_t) - \frac{\alpha}{2} \sum_{t\geqslant 0} r_{tt} x_t^2 \right] \\
\times \exp\left[- \alpha \sum_{0\leqslant t<t'} r_{tt'} x_t x_{t'} S(t+1)S(t'+1) - \alpha \sum_{0\leqslant t<t'}^{T} k_{tt'} x_{t'} S(t)S(t'+1) \right]
\end{aligned} \tag{28}
$$

where the following variables are substituted for A, i.e. $A = S(t)$ for M_t, $A = S(t)S(t')$ for $q_{tt'}$ and $A = S(t)S(t'+1)x_{t'}$ for $v_{tt'}$. Equations (23)–(25) and (28) enable us to obtain the solutions for larger time steps, beyond the second time step achieved by Gardner *et al* (see figure 3). Moreover, as we see in the next section, the perturbation theory is developed on the basis of the results presented here.

We now discuss the exact solution of the probability distribution $D_t(z)$ of the noise term $z_i(t)$. This distribution function is expressed as

$$
\begin{aligned}
D_t(z) = \Bigg\langle \mathrm{Tr} \frac{1}{N} \sum_k \delta\Bigg(z - \frac{1}{N} \xi_k^\nu \sum_{\mu\neq\nu} \xi_k^\mu \sum_{i\neq k} \xi_i^\mu S_i(0) \Bigg) \delta\Bigg(m(0) - \frac{1}{N} \sum_i \xi_i^\nu S_i(0) \Bigg) \\
\times \prod_{i,t'=0,\ldots,t-1} \theta(S_i(t'+1)\, \mathrm{sgn}[h_i(t)]) \Bigg\rangle_\xi .
\end{aligned} \tag{29}
$$

We calculate the right-hand side of this equation in a similar way to that used for equation (6). We then compare the results thus obtained to the distribution functions derived from the analysis of CS [13]. Comparisons are made as follows. First, we calculate the exact distribution at the time step t as well as the values of $m(t)$ and $r(t)$ using our exact approach. Then, we use these $m(t)$ and $r(t)$ to obtain the corresponding distribution function in the CS scheme.

In the retrieval case, both functions, our result and the formalism due to CS, coincide very well with each other as shown in figures 1(a) and (b). However, figures 1(c) and (d) show that they significantly deviate from each other even in the first time step when the retrieval fails (hereafter to be referred to as the 'non-retrieval case'). In other words, the error of the overlap at each succeeding time step in the CS theory amounts to several per cent which is not negligible when accumulated. This fact indicates that there are some cases where the assumption used in the CS theory is not necessarily fulfilled. Recently, Ozeki and Nishimori [15] arrived at a similar conclusion from their computer simulations.

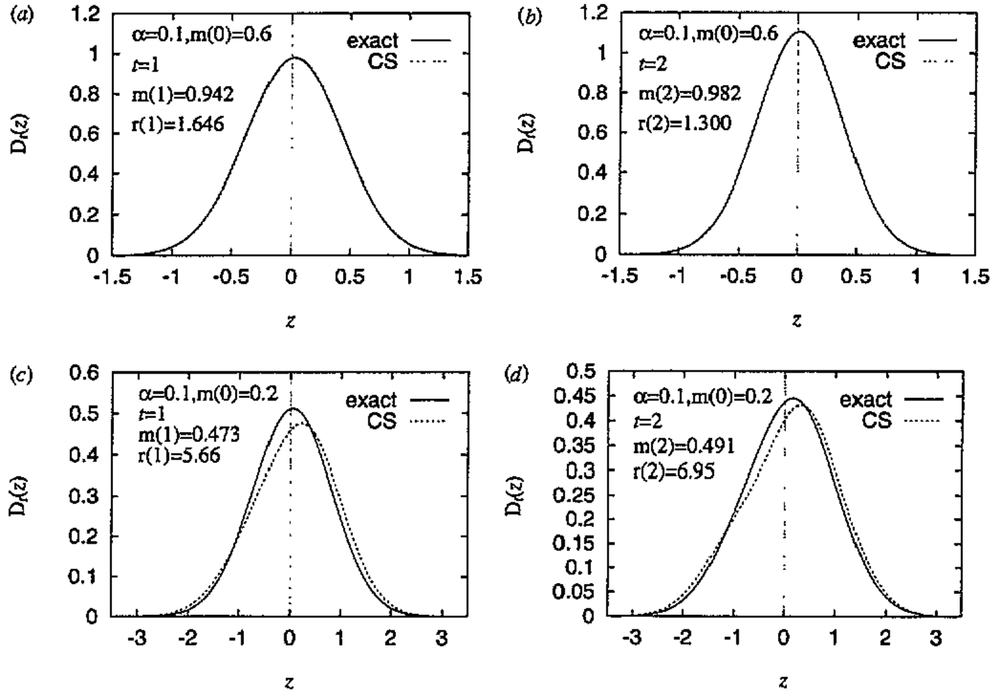

Figure 1. The probability distribution of the noise term $D_t(z)$. The exact solutions (solid curves) are compared to the functions (dashed curves) obtained by the approach of CS; (*a*) and (*b*) respectively correspond to the first and second time step in the retrieval case while (*c*) and (*d*) respectively correspond to the first and second time step in the non-retrieval case. The values of the order parameters at each time step are written in the figures.

4. Perturbation theory

4.1. Perturbation expansion

In the preceding section, we have obtained the rather simple equation (28) which defines the order parameters $m(t)$, $q_{tt'}$ and $v_{tt'}$. Although in principle it is possible to evaluate an explicit form exactly for each of these parameters from this equation, it is in practice impossible to carry out this task.

One possible way out of this difficulty is to solve equation (28) by means of some perturbation scheme. In this subsection we present a basic framework for our perturbation theory as well as a perturbation parameter used in this theory.

Let us begin with the results for parameters $k_{tt'}$ and $r_{tt'}$, obtained in the preceding section. As is seen from equations (23)–(27), each of the explicit solutions for these parameters is written in terms of power series in $v_{tt'}$:

$$k_{tt'+1} = \mathrm{i}v_{t+1t} \qquad (t = t') \tag{30}$$

$$k_{tt'+1} = \mathrm{i}(v_{t'+1t} + v_{t'+1t'}v_{t't} + v_{t'+1t'-1}v_{t'-1t} + v_{t'+1t'}v_{t't'-1}v_{t'-1t} + \cdots) \qquad (t < t') \tag{31}$$

$$r_{tt'} = 1 + 2\mathrm{i}q_{t-1t}v_{tt-1} + v_{tt-1}^2 + 2\mathrm{i}q_{t-2t}(v_{tt-2} + v_{tt-1}v_{t-1t-2}) + \cdots \qquad (t = t') \tag{32}$$

$$r_{tt'} = \mathrm{i}q_{tt'} + \mathrm{i}q_{t-1t'}v_{tt-1} + \mathrm{i}q_{tt'-1}v_{t't'-1} + r_{tt}v_{t't}\delta_{t't+1} + \cdots \qquad (t < t') \tag{33}$$

where $\delta_{t_1t_2}$ denotes the Kronecker delta symbol. Taking advantage of this fact, we assume that the values of parameters $v_{tt'}$ satisfy $v_{tt'} \ll 1$ for all t and t' up to a time step τ ($\tau \geqslant 1$,

$t' < t \leqslant \tau$). The extent of this assumption, i.e. the condition for this assumption to be valid, is clarified in what follows by the method of reduction. Substituting the expressions for $k_{tt'}$ (equations (30) and (31)) into equation (28) and expanding in terms of $v_{tt'}$, we obtain the following forms for the parameters $M_{\tau+1}q_{t\tau+1}$, $(t \leqslant \tau)$ and $v_{\tau+1t}$, $(t \leqslant \tau)$ at $\tau+1$ time step:

$$M_{\tau+1} = \mathrm{erf}\left(\frac{M_\tau}{\sqrt{2\alpha r_{\tau\tau}}}\right) + \epsilon_\tau \frac{\alpha}{\sqrt{r_{\tau\tau}}} v_{\tau 0} m(0) - \epsilon_\tau \frac{\alpha}{\sqrt{r_{\tau\tau}}} \sum_{1\leqslant t_1<\tau} v_{\tau t_1} c_{t_1-1\tau} + \mathrm{O}(v^2) \tag{34}$$

$$v_{\tau+1\tau} = \epsilon_\tau \left(\frac{1}{\sqrt{r_{\tau\tau}}} - \frac{\alpha}{\sqrt{r_{\tau\tau}}}\frac{M_\tau}{\alpha r_{\tau\tau}} v_{\tau 0} m(0) \right.$$
$$\left. + \frac{\alpha}{\sqrt{r_{\tau\tau}}} \sum_{1\leqslant t_1<\tau} v_{\tau t_1}\left[\frac{M_\tau}{\alpha r_{\tau\tau}} c_{t_1-1\tau} - \frac{r_{t_1-1\tau}}{r_{\tau\tau}}\sigma_{t_1-1\tau}\right] + \mathrm{O}(v^2)\right) \tag{35}$$

$$v_{\tau+1t} = \epsilon_\tau \left(\frac{\alpha}{\sqrt{r_{\tau\tau}}}\sigma_{t\tau} v_{\tau t+1} + \mathrm{O}(v^2)\right) \qquad (t < \tau) \tag{36}$$

where $\mathrm{erf}(x)$ denotes the error function ($\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{d}t\, \mathrm{e}^{-t^2}$) and the definitions for ϵ_{t_1}, $c_{t_1t_2}$ and $\sigma_{t_1t_2}$ are given by

$$\epsilon_{t_1} = \sqrt{\frac{2}{\pi\alpha}} \exp\left(-\frac{M_{t_1}^2}{2\alpha r_{t_1t_1}}\right) \tag{37}$$

$$c_{t_1t_2} = -\mathrm{erf}\left(\frac{\Delta_{t_1t_2}}{\sqrt{2\alpha R_{t_1t_2}}}\right) \tag{38}$$

$$\sigma_{t_1t_2} = \sqrt{\frac{2}{\pi\alpha R_{t_1t_2}}} \exp\left(-\frac{\Delta_{t_1t_2}^2}{2\alpha R_{t_1t_2}}\right) \tag{39}$$

where $\Delta_{t_1t_2}$ and $R_{t_1t_2}$ are defined by

$$\Delta_{t_1t_2} \equiv M_{t_1} - \frac{r_{t_1t_2}}{r_{t_2t_2}} M_{t_2} \tag{40}$$

$$R_{t_1t_2} \equiv \frac{r_{t_1t_1} r_{t_2t_2} - r_{t_1t_2}^2}{r_{t_2t_2}}. \tag{41}$$

We see from equation (32) that $r_{\tau\tau} = 1 + \mathrm{O}(v)$. Using this relation, equations (35) and (36) are written as

$$v_{\tau+1\tau} = \epsilon_\tau (1 + \mathrm{O}(v)) \tag{42}$$

$$v_{\tau+1t} = \epsilon_\tau (\mathrm{O}(v)). \tag{43}$$

Therefore, our requirement $v_{tt'} \ll 1$ is also valid at the time step $\tau+1$ if $v_{tt'} \ll 1$ for $t' < t \leqslant \tau$ and the condition

$$v_{\tau+1\tau} \sim \epsilon_\tau = \sqrt{\frac{2}{\pi\alpha}} \exp\left(-\frac{M_\tau^2}{2\alpha r_{\tau\tau}}\right) \ll 1 \tag{44}$$

is satisfied. In other words, our theory is reliable as long as the relation

$$M_\tau \gg \sqrt{2\alpha r_{\tau\tau}} \tag{45}$$

is fulfilled.

In the retrieval case the overlap $m(t) = M_t$ increases with the time step, whereas the non-retrieval parameter $r(t) = r_{tt}$ tends to decrease. Therefore, condition (45) is valid in this case if it is satisfied at the outset. On the other hand, it is not fulfilled in the non-retrieval case. Thus, the extent of our theory is found in the theoretical scheme itself.

We notice here that there are several small parameters in the right-hand side of equations (34)–(36), i.e. ϵ_τ and $v_{\tau t}$ for $t < \tau$. In order to make the expansion more systematic and to make our theory clearer, we express the parameters $v_{\tau t}$ in these equations in terms of ϵ. In other words, we adopt parameter ϵ_t as a new perturbation parameter in our theory. From equations (42) and (43) and the relation $\epsilon_\tau - \epsilon_{\tau-1} < \mathrm{O}(\epsilon_\tau)$, equations (34), (35) and (36) are written as

$$\begin{aligned} M_{\tau+1} &= \mathrm{erf}\left(\frac{M_\tau}{\sqrt{2\alpha r_{\tau\tau}}}\right) - \alpha\frac{c_{\tau-2\tau}}{r_{\tau\tau}}\epsilon_\tau\epsilon_{\tau-1} + \mathrm{O}(\epsilon^3) \\ &= \mathrm{erf}\left(\frac{M_\tau}{\sqrt{2\alpha r_{\tau\tau}}}\right) - \alpha c_{\tau-2\tau}\epsilon_\tau^2 + \mathrm{O}(\epsilon^3) \end{aligned} \tag{46}$$

$$\begin{aligned} v_{\tau+1\tau} &= \frac{\epsilon_\tau}{\sqrt{r_{\tau\tau}}} + \alpha\left[\frac{M_\tau}{\alpha r_{\tau\tau}}c_{\tau-2\tau} - \frac{r_{\tau-2\tau}}{r_{\tau\tau}}\sigma_{\tau-2\tau}\right]\frac{\epsilon_\tau\epsilon_{\tau-1}}{r_{\tau\tau}} + \mathrm{O}(\epsilon^3) \\ &= \epsilon_\tau + [c_{\tau-2\tau} - \alpha\sigma_{\tau-2\tau} - q_{\tau-1\tau}]\epsilon_\tau^2 + \mathrm{O}(\epsilon^3) \end{aligned} \tag{47}$$

$$v_{\tau+1t} = \mathrm{O}(\epsilon^3) \qquad (t < \tau). \tag{48}$$

Note that when condition (45) is satisfied, the first term in equation (46) is written as the sum of an infinite number of the first-order terms in ϵ_τ:

$$\mathrm{erf}\left(\frac{M_\tau}{\sqrt{2\alpha r_{\tau\tau}}}\right) = 1 - \sqrt{\frac{\alpha}{2}}\left(\frac{\sqrt{2\alpha r_{\tau\tau}}}{M_\tau}\right)\epsilon_\tau + \frac{1}{2}\sqrt{\frac{\alpha}{2}}\left(\frac{\sqrt{2\alpha r_{\tau\tau}}}{M_\tau}\right)^3\epsilon_\tau + \cdots. \tag{49}$$

Therefore, we leave this term as it stands in expansion (46) without expanding it in ϵ_τ.

The perturbation expansion of the remaining parameters, i.e. $q_{tt'}$ and $r_{tt'}$, are now discussed. We derive the following expression for $q_{tt'}$, in a similar way as we derived equations (34)–(36):

$$\begin{aligned} \mathrm{i}q_{t\tau+1} = &\int_{-\infty}^{\infty}\mathrm{D}y\,\mathrm{sgn}\left(M_\tau + \sqrt{2\alpha r_{\tau\tau}}y\right)\mathrm{erf}\left(\frac{M_\tau + \sqrt{2\alpha}r_{t-1\tau}y/\sqrt{r_{\tau\tau}}}{\sqrt{2\alpha R_{t-1\tau}}}\right) \\ &-\epsilon_\tau\frac{\alpha}{\sqrt{r_{\tau\tau}}}v_{\tau 0}m(0)c_{t-1\tau} + \epsilon_\tau\frac{\alpha}{\sqrt{r_{\tau\tau}}}v_{\tau t} \\ &+\epsilon_\tau\frac{\alpha}{\sqrt{r_{\tau\tau}}}\sum_{1\leqslant t_1<\tau, t_1\neq t} v_{\tau t_1}\left[\int_{-\infty}^{\infty}\mathrm{D}y\,\mathrm{sgn}\left(\Delta_{t-1\tau} + \sqrt{2\alpha R_{t-1\tau}}y\right)\right. \\ &\left.\times\,\mathrm{erf}\left(\frac{\Delta_{t_1-1\tau} + \sqrt{2\alpha}R^{(\tau)}_{t_1-1t-1}y/\sqrt{R_{t-1\tau}}}{\sqrt{2\alpha\zeta^{(\tau)}_{t_1-1t-1}}}\right)\right] - \epsilon_{t-1}\frac{\alpha}{\sqrt{r_{t-1t-1}}}v_{t-10}m(0)c_{\tau t-1} \\ &+\epsilon_{t-1}\frac{\alpha}{\sqrt{r_{t-1t-1}}}\sum_{1\leqslant t_1<t-1} v_{t-1t_1}\left[\int_{-\infty}^{\infty}\mathrm{D}y\,\mathrm{sgn}\left(\Delta_{\tau t-1} + \sqrt{2\alpha R_{\tau t-1}}y\right)\right. \\ &\left.\times\,\mathrm{erf}\left(\frac{\Delta_{t_1-1t-1} + \sqrt{2\alpha}R^{(t-1)}_{t_1-1\tau}y/\sqrt{R_{\tau t-1}}}{\sqrt{2\alpha\zeta^{(t-1)}_{t_1-1\tau}}}\right)\right] + \mathrm{O}(v^2) \qquad (t \leqslant \tau) \end{aligned} \tag{50}$$

where $\int\mathrm{D}y = \int\frac{\mathrm{d}y}{\sqrt{2\pi}}\mathrm{e}^{-y^2}$ and the definitions for $R^{(t_3)}_{t_1t_2}$ and $\zeta^{(t_3)}_{t_1t_2}$ are given by

$$R^{(t_3)}_{t_1t_2} \equiv \frac{r_{t_3t_3}r_{t_1t_2} - r_{t_1t_3}r_{t_2t_3}}{r_{t_3t_3}} \tag{51}$$

$$\zeta^{(t_3)}_{t_1t_2} \equiv \frac{R_{t_1t_1}R_{t_2t_2} - (R^{(t_3)}_{t_1t_2})^2}{R_{t_2t_2}}. \tag{52}$$

Using equations (32), (33), (42), (43) and (50) and the definition of $R_{t_1t_2}$, equation (41), we obtain the relation in a self-consistent manner:

$$1 - \mathrm{i}q_{\tau\tau+1} < \mathrm{O}(\epsilon) \tag{53}$$

$$R_{\tau\tau+1} < \mathrm{O}(\epsilon). \tag{54}$$

From equations (32), (47), (48) and (53), it is shown that

$$r_{\tau+1\tau+1} = 1 + 2\epsilon_\tau + (1 + 2c_{\tau-2\tau} - 2\alpha\sigma_{\tau-2\tau})\epsilon_\tau^2 + \mathrm{O}(\epsilon^3). \tag{55}$$

Finally, we have to consider the time dependences of $c_{\tau-2\tau}$ and $\sigma_{\tau-2\tau}$ which appear in the second-order terms of equations (46) and (55). It is shown from the definitions of $c_{\tau-2\tau}$ and $\sigma_{\tau-2\tau}$ (equations (38) and (39)) that these quantities are not simply expressed as the power series in ϵ. In order to overcome this difficulty we study the behaviour of $c_{\tau-2\tau}$ and $\sigma_{\tau-2\tau}$ with the help of computer simulations. The results of our simulations for $c_{\tau-2\tau}$ and $\sigma_{\tau-2\tau}$ are shown, respectively, in figures 2(*a*) and (*b*), starting from four different initial conditions for $m(0)$ in the case of $\alpha = 0.1$ and $N = 2500$. For each value of the initial overlap the retrieval is successful. From these figures we observe that $c_{\tau-2\tau}$ approaches zero and $\sigma_{\tau-2\tau}$ diverges to ∞ when $t \to \infty$ in the retrieval case. However, the value of the non-retrieval parameter $r_{\tau\tau}$ is finite from equation (5). Therefore, the term with $\sigma_{\tau-2\tau}$ is supposed to be cancelled out by the higher-order terms in ϵ_τ and, hence, we introduce an assumption that this term can be omitted. Substituting $M_\tau = m(t)$, $r_{\tau\tau} = r(t)$, $c_{\tau-2\tau} = c(t)$ and $\epsilon_\tau = \epsilon(t)$ into equations (46) and (55) we obtain the final forms for the important order parameters as

$$m(t+1) = \mathrm{erf}\left(\frac{m(t)}{\sqrt{2\alpha r(t)}}\right) - \alpha c(t)\epsilon^2(t) \tag{56}$$

$$r(t+1) = 1 + 2\epsilon(t) + (1 + 2c(t))\epsilon^2(t) \tag{57}$$

with the time-dependent perturbation parameter

$$\epsilon(t) = \sqrt{\frac{2}{\pi\alpha}}\exp\left(-\frac{m^2(t)}{2\alpha r(t)}\right). \tag{58}$$

In following sections we show the results of our theory based on equations (56)–(58).

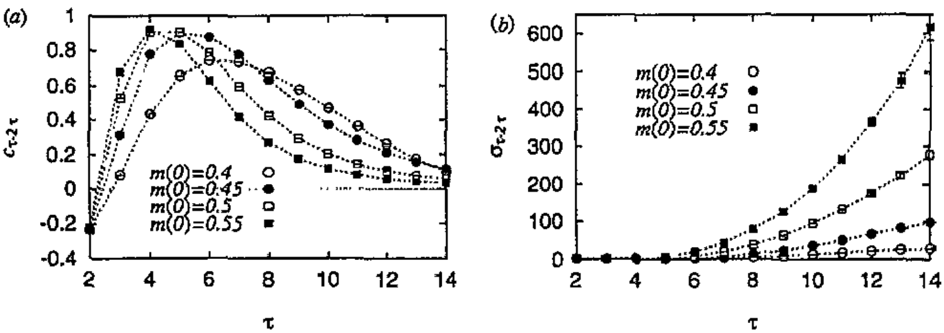

Figure 2. Time development of $c_{\tau-2\tau}$ and $\sigma_{\tau-2\tau}$. The results of our simulations for $c_{\tau-2\tau}$ and $\sigma_{\tau-2\tau}$ are respectively depicted in (*a*) and (*b*) for the case of $\alpha = 0.10$ and $N = 2500$. Different curves in each figure correspond to the different initial overlaps $m(0)$.

4.2. Approximation up to first order

Keeping the term up to first order in $\epsilon(t)$, we obtain from equations (56) and (57)

$$m(t+1) = \operatorname{erf}\left(\frac{m(t)}{\sqrt{2\alpha r(t)}}\right) \tag{59}$$

$$r(t+1) = 1 + 2\epsilon(t). \tag{60}$$

We compare our results with the recursion relations obtained by the AM theory, the latter being expressed as

$$m(t+1) = \operatorname{erf}\left(\frac{m(t)}{\sqrt{2\alpha r(t)}}\right) \tag{61}$$

$$r(t+1) = 1 + 2\frac{m(t)m(t+1)}{\sqrt{r(t)}}\epsilon(t) + \epsilon^2(t). \tag{62}$$

In our theory, as we have seen in the preceding subsection, $m(t) = 1 - \mathrm{O}(\epsilon(t))$ and $r(t) = 1 + \mathrm{O}(\epsilon(t))$. Therefore, the AM theory is perfectly comparable with our perturbation theory up to first order in $\epsilon(t)$.

In addition, our calculations for the distribution functions $D_t(z)$ of the noise terms, which are performed in a similar way to the calculations for the parameters M_τ or $v_{\tau+1\tau}$ in the preceding section, give the Gaussian distributions as the leading-order approximations:

$$D_t(z) = \sqrt{\frac{1}{2\pi\alpha r(t)}} \exp\left(-\frac{z^2}{2\alpha r(t)}\right). \tag{63}$$

Thus, our perturbation theory up to first order supports the AM theory and provides the basis for the assumption used in the AM theory.

4.3. Approximation up to second order

We now turn to the results of our perturbation theory up to second order in $\epsilon(t)$; namely, we take all the terms in equations (56) and (57) into account.

It is obvious that the recursion relations of the AM theory, equations (61) and (62), do not coincide with those of our theory when the second-order terms are taken into account. The noise distributions $D_t(z)$ also deviate from the Gaussian forms and are written as

$$D_t(z) = \sqrt{\frac{1}{2\pi\alpha r(t)}} \exp\left(-\frac{z^2}{2\alpha r(t)}\right)(1 - c(t)\epsilon(t)z). \tag{64}$$

We now show from equations (56) and (57) that the value of parameter $r(t)$ is determined from the value of the overlap $m(t)$ at the same time step. In other words, the trajectories in the $m(t)$–$r(t)$ plane of the system with various initial conditions fall onto one master curve, which is described as

$$r(t) = r(m(t)) = \left(1 + \sqrt{\frac{2}{\pi\alpha}} \exp(-[\operatorname{erf}^{-1}(m(t))]^2)\right)^2 (+\mathrm{O}(\epsilon^3(t))). \tag{65}$$

The derivation of this equation is given in appendix A. Note that equation (65) is valid irrespective of the value of $c(t)$. Relation (65), when plotted in the $m(t)$–$r(t)$ plane, was called the freezing line in the original articles of Coolen and Sherrington [13]. They define the freezing line as a line *outside* the region to which an exponentially small number of microscopic configurations $\{S_i\}$ belong. (Here, 'outside' the region means the region with larger values of $m(t)$ and $r(t)$.) As was shown in the erratum to their paper [16], the *true*

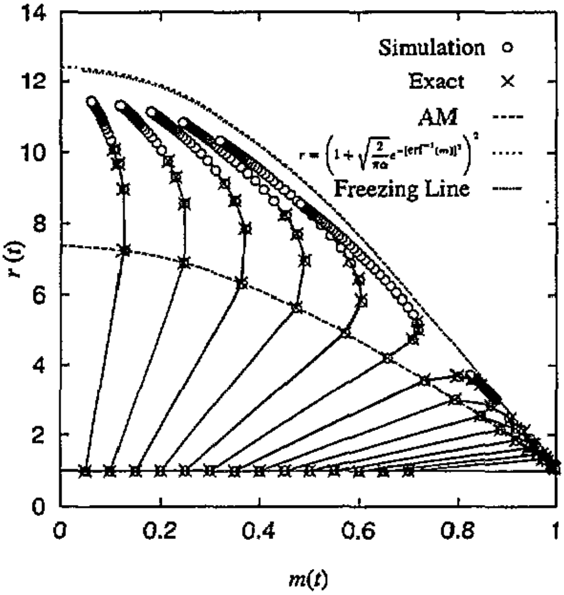

Figure 3. The trajectories in the m–r plane. The open circles with lines represent the trajectories obtained from our simulations starting from 14 different initial conditions for $m(0)$; i.e. $0.05 \leqslant m(t) \leqslant 0.7$, and $r(0) = 1$, where $\alpha = 0.1$ and $N = 2500$. The crosses represent the exact trajectories. The dotted curve is the freezing line. The short dashes represents the curve expressed by equation (65). The trajectories predicted by the AM theory fall on the long dashed curve.

freezing line is not exactly expressed by equation (65). However, it is known [17] that the deviation of the freezing line from equation (65) is negligibly small (see figure 3).

Our theory predicts that the time evolutions of the order parameters take place at the edge of the region within which almost all of the microscopic configurations are involved. On the other hand, the trajectories according to the AM theory in the $m(t)$–$r(t)$ plane converge onto a different curve. In the following section we show that the results of our computer simulations support the prediction of our perturbation theory, rather than that of the AM theory.

As the time step proceeds, the value of $c(t)$ becomes negligibly small as is mentioned in section 4.1. In this case, we obtain the recursion relation from equation (56) including the overlap only:

$$m(t+1)\,\mathrm{erf}\left(\frac{m(t)}{\sqrt{2\alpha}\left(1+\sqrt{\frac{2}{\pi\alpha}}\mathrm{e}^{-[\mathrm{erf}^{-1}(m(t))]^2}\right)}\right). \tag{66}$$

The stationary value of the overlap is given by putting $m(t+1) = m(t)$ in equation (66). *It is interesting to note that the resulting equation for the stationary state is the same, up to* the second order in $\epsilon(t)$, as that for the equilibrium state obtained by Amit *et al* [18] using the replica symmetric analysis [1]. The maximal storage capacity α_c of the Little–Hopfield model obtained from equation (66) is almost equal to the well known value

$$\alpha_c \simeq 0.138. \tag{67}$$

Recently, the equilibrium state of the Little–Hopfield model has been analysed [19, 20] in the framework of the replica-symmetry-breaking (RSB) ansatz [21]. It is quite an interesting problem to clarify the relation between the results of these studies of equilibrium states and of our theory on the dynamics.

5. Results of our simulations

In order to clarify the validity of our theoretical results explained in the preceding sections, we show in this section the results of our simulations of the Little–Hopfield model.

In figure 3 the trajectories in the $m(t)$–$r(t)$ plane are shown for 14 different initial conditions in the case of $\alpha = 0.1$ and $N = 2500$. The results obtained from our simulations are described by open circles. The crosses represent the results evaluated from our exact solutions (demonstrated in section 3) up to the fourth step. The perfect agreements between the trajectories of our simulations and of the exact solutions indicate that the finite-size effects in our simulations are negligibly small.

The dotted curve in this figure represents the freezing line which is determined from the equations due to Coolen and Sherrington [16]. On the other hand, the short dashes denote the curve described by equation (65). Note that it is difficult to find the differences between these curves. It is clearly seen from this figure that the trajectories for the retrieval cases converge to the freezing line as our theory predicts (in section 4). On the other hand, the AM theory predicts that the trajectories fall on the curve with long dashes which is also plotted in this figure. Although the AM theory was believed to be correct in the retrieval case, it is apparent that this theory, which is consistent with our theory only within the extent of the first-order approximation, fails to describe the above-mentioned behaviour in the $m(t)$–$r(t)$ plane.

Thus, our simulations support the prediction of our perturbation theory and, consequently, support our theory itself. A remarkable point to make is the fact that both our theory and our simulations indicate the obvious relationship between the long-time trajectories and the freezing line. The relationship is especially clear in the retrieval case. Since the freezing line is defined as a kind of boundary between the allowed and non-allowed regions for the existence of microscopic configurations, the convergence of the long-time trajectories towards this line suggests that either retrieval or non-retrieval is pursued along the edge of reality.

6. Summary and discussion

In this paper we study the dynamical behaviour of the Little–Hopfield model. Our present work is summarized as follows.

First, we improve the exact approach.

(i) The explicit solutions for some of the order parameters, including the non-retrieval parameter $r(t)$, are obtained at arbitrary time steps.

(ii) The exact solutions up to the fourth step are evaluated (figure 3).

(iii) The probability distributions $D_t(z)$ of the noise term are calculated exactly. It is shown that there are some cases in which the assumption used by Coolen and Sherrington is not necessarily fulfilled.

Second, on the basis of the results listed in [1], we propose a perturbation theory.

(i) The equations for the time developments of the order parameters, obtained in the approximation up to first order, have a form consistent with those of the theory by Amari and Maginu [7]. However, a discrepancy between these theories arises when we include the second-order terms in the expansions.

(ii) Our theory predicts, within the approximation up to second order, that the trajectories in the $m(t)$–$r(t)$ plane converge to the freezing line in the retrieval case. This prediction is supported by the results of our computer simulations.

(iii) From the second-order approximation, we show that the stationary state is almost the same as the equilibrium state obtained by the replica symmetric analysis.

The perturbation theory proposed in this paper will also be applicable, by choosing an appropriate perturbation parameter, to the present model with finite temperatures and other fully connected networks such as the Sherrington and Kirkpatrick model [1, 22] for a spin glass. It is extremely interesting and important to compare the dynamical properties of these models by this general approach.

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Appendix A. Derivation of equation (65)

From equation (57) we obtain

$$\begin{aligned}\mathrm{erf}^{-1}[m(t+1)] &= \mathrm{erf}^{-1}\left[\mathrm{erf}\left(\frac{m(t)}{\sqrt{2\alpha r(t)}}\right) - \alpha c(t)\epsilon^2(t) + \mathrm{O}(\epsilon^3(t))\right] \\ &= \frac{m(t)}{\sqrt{2\alpha r(t)}} - \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{erf}^{-1}(x)\bigg|_{x=\mathrm{erf}(m(t)/\sqrt{2\alpha r(t)})} \alpha c(t)\epsilon^2(t) + \mathrm{O}(\epsilon^2(t)) \\ &= \frac{m(t)}{\sqrt{2\alpha r(t)}} - \sqrt{\frac{\alpha}{2}}c(t)\epsilon(t) + \mathrm{O}(\epsilon^2(t))\end{aligned} \tag{A1}$$

where the relation

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{erf}^{-1}(x) = \frac{\sqrt{\pi}}{2}\exp[(\mathrm{erf}^{-1}(x))^2]$$

has been used. Therefore, $\epsilon(t)$ is expressed as

$$\begin{aligned}\epsilon(t) &= \sqrt{\frac{2}{\pi\alpha}}\exp\left[-\left(\mathrm{erf}^{-1}(m(t+1)) + \sqrt{\frac{\alpha}{2}}c(t)\epsilon(t) + \mathrm{O}(\epsilon^2(t))\right)^2\right] \\ &= \sqrt{\frac{2}{\pi\alpha}}\exp[-(\mathrm{erf}^{-1}(m(t+1)))^2](1 + c(t)\epsilon(t) + \mathrm{O}(\epsilon^2(t))).\end{aligned} \tag{A2}$$

Substituting equation (A2) into equation (57), we obtain expression (65) in section 4.

References .

[1] Mezard M, Parisi G and Virasoro M A 1987 *Spin Glass Theory and Beyond* (Singapore: World Scientific)
Fischer K H and Hertz J A 1991 *Spin Glasses* (Cambridge: Cambridge University Press)

[2] Amit D J 1989 *Modeling Brain Function* (Cambridge: Cambridge University Press)
Domany E, van Hemmen J L and Schulten K (ed) 1991 *Models of Neural Networks* (Berlin: Springer)
Hertz J, Krogh A and Palmer R P 1991 *Introduction to the Theory of Neural Computation* (Reading, MA: Addison-Wesley)

[3] Little W A 1974 *Math. Biosci.* **19** 101

[4] Hopfield J J 1982 *Proc. Nat. Acad. Sci., USA* **79** 2554

[5] Gardner E, Derrida B and Mottishaw P 1987 *J. Physique* **48** 741

[6] Rieger H, Schreckenberg M and Zittartz J 1988 *J. Phys. A: Math. Gen.* **21** L263; *Z. Phys.* B **72** 323

[7] Amari S and Maginu K 1988 *Neural Networks* **1** 63

[8] Horner H, Bormann D, Frick M, Kinzelbach H and Schmidt A 1989 *Z. Phys.* B **76** 381
[9] Zagrebnov Z Z A and Chvyrov A S 1989 *Sov. Phys.–JETP* **68** 153
[10] Eissfeller H and Opper M 1992 *Phys. Rev. Lett.* **68** 2094
[11] Shukla P 1993 *J. Stat. Phys.* **71** 705
[12] Nishimori H and Ozeki T 1993 *J. Phys. A: Math. Gen.* **26** 859
[13] Coolen A C C and Sherrington D 1993 *Phys. Rev. Lett.* **71** 3886; 1994 *Phys. Rev.* E **49** 1921
[14] Okada M 1995 A hierarchy of macrodynamical equations for associative memory *Neural Networks* in press
[15] Ozeki T and Nishimori H 1994 *J. Phys. A: Math. Gen.* **27** 7061
[16] Coolen A C C and Sherrington D 1994 *Phys. Rev.* E **49** 5906
[17] Coolen A C C and Sherrington D 1994 *J. Phys. A: Math. Gen.* **27** 7687
[18] Amit D J, Gutfreund H and Somplinsky H 1985 *Phys. Rev. Lett.* **55** 1530; 1987 *Ann. Phys., NY* **173** 30
[19] Steffan H and Kühn R 1994 Replica symmetry breaking in attractor neural network models *Z. Phys.* B submitted
[20] Tokita K 1994 *J. Phys. A: Math. Gen.* **27** 4413
[21] Parisi G 1980 *J. Phys. A: Math. Gen.* **13** L115; *J. Phys. A: Math. Gen.* **13** 1101; 1983 *Phys. Rev. Lett.* **50** 1946
[22] Sherrington D and Kirkpatrick S 1975 *Phys. Rev. Lett.* **35** 1792